Chancellor's Honors Program Projects   Supervised Undergraduate Student Research and Creative Work

5-2019

Constructions of Hadamard Matrices

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April 27, 2019

Introduction

In 1933, R. E. A. C. Paley's paper *On Orthogonal Matrices* was published. He introduced lemmas that contain step-by-step methods to create Hadamard matrices. After searching the internet, we could not find any evidence that someone has taken the time to compile a visual library of all the matrices that Paley's lemmas can be used to construct. With this in mind, we decided to start a visual library that contained a Hadamard matrix for every order less than 100. *On Orthogonal Matrices* contains the methods to construct every matrix of order less than 100 except for 92, but some of the constructions require knowledge of topics with which I am unfamiliar. Therefore, we decided to use Paley's three lemmas that only require previous knowledge of quadratic residues mod p, as well as Sylvester's Construction and Williamson matrices to complete this visual library.

We begin the paper by stating the definition of a real Hadamard matrix and proving two important theorems that pertain to Hadamard matrices. Then the various constructions are introduced in this order: Sylvester's Construction, Paley's Construction, and the Williamson Construction. In each section, the lemmas and theorems associated with each construction are stated along with the R code that was used to create one of matrices. This R code can be easily adapted to construct Hadamard matrices of higher orders if the reader wishes to do so. It is important that the R code is only used to create matrices that meet the assumptions of each construction. The names of the libraries in R that are needed for the various commands are all in the code.

Contents

Hadamard Matrices

Hadamard Matrix Definition A *real square matrix H is said to be a real Hadamard matrix if all entries of H have absolute value 1 and all rows of H are mutually orthogonal. That is a real square matrix $H = (h_{ij})$ of order n is a Hadamard if $|h_{ij}| = 1$ for all $i, j \leq n$ and for each pair of distinct rows h_i and h_j of H, $h_i \bullet h_j = 0$.*

Examples

$$n = 2: \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad n = 4: \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

To check that each pair of distinct rows are mutually orthogonal, we need to check that the dot product of each row with every other row is zero. For n = 2, the computation is below. Note that • is used to denote the dot product while · is used to denote multiplication.

$$(1,1) \bullet (1, -1) = 1 \cdot 1 + 1 \cdot (-1) = 0$$

Therefore, each pair of distinct rows in this matrix of order 2 are mutually orthogonal and the matrix consists of 1 and -1, therefore it is a Hadamard matrix. Similarly, it can be shown that the matrix above of order 4 is a Hadamard matrix.

Below is a theorem along with its proof from [1] that makes it easy to use software to check whether a matrix is a Hadamard matrix. In the examples section towards the end of this paper, this computation using Theorem 1 is shown next to each matrix to prove that the matrix is a Hadamard matrix.

Theorem 1 *Let $X = (x_{ij})$ be an n x n real matrix whose entries satisfy $|x_{ij}| \leq 1$ for all i, j. Then $|\det(X)| \leq n^{\frac{n}{2}}$. Equality holds if and only if H is a Hadamard matrix.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the rows of a real matrix, X, satisfying the conditions above. The rows of X form the sides of a parallelepiped. Therefore, we know that
$$\det(X) \leq |x_1||x_2| \cdots |x_n|$$
where $|x_i| = (x_{i1}^2 + x_{i2}^2 + \cdots + x_{in}^2)^{\frac{1}{2}}$. Because all entries of X satisfy $|x_{ij}| \leq 1$, then
$$|x_i| \leq (n \cdot 1)^{\frac{1}{2}} = n^{\frac{1}{2}}.$$
Hence, $|\det(X)| \leq |x_1||x_2| \cdots |x_n| \leq n \cdot n^{\frac{1}{2}} = n^{\frac{n}{2}}$, so $|\det(X)| \leq n^{\frac{n}{2}}$.

For the second part of the proof, assume that $|\det(X)| = n^{\frac{n}{2}}$. Therefore,
$$(x_{11}^2 + \cdots + x_{1n}^2)^{\frac{1}{2}} \cdot (x_{21}^2 + \cdots + x_{2n}^2)^{\frac{1}{2}} \cdots (x_{n1}^2 + \cdots + x_{nn}^2)^{\frac{1}{2}} = n^{\frac{n}{2}},$$

which means that $|x_{ij}| = 1$ for all entries of X. We also know that $\det(X) \leq |x_1||x_2| \cdots |x_n|$ if and only if each pair of distinct rows are mutually orthogonal, therefore each pair of distinct rows are mutually orthogonal. We have a matrix where all entries are 1 or -1 and where all the rows are mutually orthogonal, therefore X is a Hadamard matrix.

If you assume that X is a Hadamard matrix, then using the same logic as above it follows that $|\det(X)| = n^{\frac{n}{2}}$.

To know for which orders a Hadamard matrix exists we use the theorem below and include its proof from [2].

Theorem 2 *Let m be the order of a Hadamard matrix, apart from when m = 1 or m = 2, it is necessary that m should be divisible by 4.*

Proof. Let $H = (h_{ij})$ be a Hadamard matrix, $(0 \leq i \leq m - 1, (0 \leq j \leq m - 1; m \geq 3)$. Then,

$$\sum_{j=0}^{m-1} (h_{1j} + h_{2j})(h_{1j} + h_{3j}) = \sum_{j=0}^{m-1} h_{1j}^2 = m$$

and

$$(h_{1j} + h_{2j})(h_{1j} + h_{3j}) = 4 \text{ or } 0.$$

We see that it must be 4 for $\frac{1}{4}m$ values of j and that m must be divisible by 4.

Immediately after this proof, Paley notes that "it seems probable that, whenever m is divisible by 4, it is possible to construct an orthogonal matrix of order m composed of ± 1, but the general theorem has every appearance of difficulty". The Hadamard Conjecture states that if n is a multiple of 4, then there exists an n x n Hadamard matrix, but this has never been proven.

Sylvester's Construction

**Theorem 3** *If $n = 2^k$ then we can construct an n x n Hadamard matrix H_n recursively,*

$$H_{2n} = \begin{bmatrix} Hn & Hn \\ Hn & -Hn \end{bmatrix}.$$

This theorem follows from the fact that the Kronecker product of two Hadamard matrices is a Hadamard matrix. Therefore, matrices of order 2, 4, 8, 16, 32, and 64 can be constructed using this theorem. These matrices are shown at the end of this paper and are in blue. Below is the code used to construct the Hadamard matrix of order 64 using Sylvester's construction. Note that the matrix of 32 had to have been constructed previously.

```r
# n = 64
#Binds together the sylv 32 matrix by the rows
sylv64 = rbind(sylv32,sylv32)
#Binds together the matrix above by columns
sylv64 = cbind(sylv64,sylv64)
#This for loop negates the lower right square of the matrix that consists of rows 33-64 and columns 33-64
for (i in 33:64) {
  for (j in 33:64) {
    sylv64[i,j] = -(sylv64[i,j])
  }
}
library(qdap)
qheat(t(data.frame(sylv64)), by.column=NULL, high="blue", low = "white", text.color =NA,
    grid="black")
```
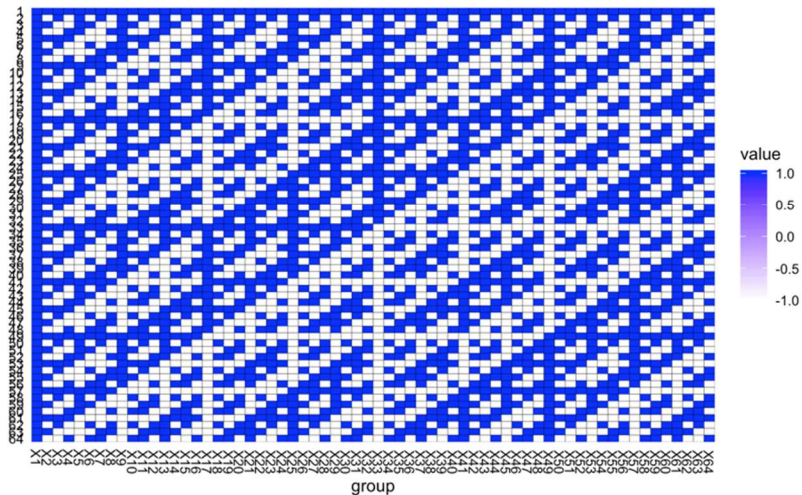
Paley's Construction

The three lemmas below and their proofs were found in [2].

Lemma 1 *Let m be of the form p + 1, where $p \equiv 3 (mod\ 4)$ and p is prime. Then we can construct a Hadamard matrix of order m.*

Proof. Let $\chi(n)$ denote the Legendre symbol (n/p). We write

$$h_{ij} = +1 \qquad (i = 0\ or\ j = 0)$$
$$h_{ij} = \chi(j - i) \quad (1 \leq i \leq p, 1 \leq j \leq p, i \neq j)$$
$$h_{ii} = -1 \qquad (1 \leq i \leq p)$$

If i_1 and i_2 are different and both are greater than 0, we have

$$\sum_{j=0}^{p}(h_{i_1 j})(h_{i_2 j}) = (h_{i_1 i_1})(h_{i_2 i_1}) + (h_{i_1 i_2})(h_{i_2 i_2}) + (h_{i_1 0})(h_{i_2 0}) + \sum_{j=1}^{p}\chi(j - i_1)\chi(j - i_2).$$

The first two terms on the right-hand side of this equation are
$$-\chi(i_1 - i_2) - \chi(i_2 - i_1)$$
and are equal and opposite since $p \equiv 3 (mod\ 4)$, therefore $\chi(-1) = -1$.
The third term on the right-hand side of the equation is +1 and the last sum is known to be -1. Therefore, we have that

$$\sum_{j=0}^{p}(h_{i_1 j})(h_{i_2 j}) = 0 \quad (0 \neq i_1 \neq i_2).$$

Now, if $i \neq 0$,

$$\sum_{j=0}^{p}(h_{0j})(h_{ij}) = \sum_{j=0}^{p}h_{ij} = h_{ii} + h_{i0} + \sum_{j=1}^{p}\chi(j - i) = 0.$$

Therefore, our matrix is orthogonal.
Using this lemma, Paley constructed the Hadamard matrix of order 12 shown on the next page.

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1
\end{vmatrix}$$

A note about the Legendre Symbol, χ, to show why this works.

Let p be a prime. For p > 2, we define $\chi(\frac{a}{p})$ to equal 0 if $a \equiv 0 \pmod p$, to equal -1 if $a \equiv x^2 \pmod p$ for some $x \in Z_p \backslash \{0\}$, and equal to 1 otherwise. Therefore, the Legendre Symbol is keeping track of all the nonzero square residues mod p.

For example, let p = 7 and look at the squares in Z_7.

$\{0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2\} \pmod 7 = \{0, 1, 4, 2, 2, 4, 1\}$, remove 0 and we get {1, 2, 4}.

So, $\chi(\frac{1}{7}) = \chi(\frac{2}{7}) = \chi(\frac{4}{7}) = -1$ and $\chi(\frac{3}{7}) = \chi(\frac{5}{7}) = \chi(\frac{6}{7}) = 1$.

Therefore, to create the 8 x 8 Hadamard matrix, we first create a 7 x 7 circulant matrix denoted as Q. The first row of Q will have the zeroth entry as +1, and then the first, second, ... , and sixth entry correspond to the Legendre symbols above. The first, second, and fourth entry will be -1, while the third, fifth, and sixth entry will be 1. Because Q is circulant, the rest of the matrix is easily constructed. For the 8 x 8 Hadamard matrix, the first row will be all +1, the first column will be all -1 except for the first entry, and then Q fills in the rest. This matrix is shown below.

$$\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1
\end{vmatrix}$$

Below is the code used to create 12 x 12 Hadamard matrix from Lemma 1 and comments were added to aid the reader.

```r
```{r}
x = c()
#Finding the elements which are squares mod 11
for (i in 0:10) {
  x[i+1] = i^2%%11
}
#Taking out repeated values in x and removing 0
x = sort(unique(x))
x = x[!x %in% c(0)]
#Adding one to all values in x to match the positions in the matrix
x = x + 1
#Creating the empty 12x12 matrix
had12 = matrix(0, nrow = 12, ncol = 12)
#Filling in the first row of the matrix with +1
for (i in 1:12) {
  had12[1,i] = 1
}
#Filling in the first column of the matrix with -1 (except the first entry)
for (j in 2:12) {
  had12[j,1] = -1
}
#Making the entries of the diagonal +1
for (r in 1:12) {
  had12[r,r] = 1
}
```
```

```r
```{r}
#Creating Q
Q = matrix(0, nrow = 11, ncol = 11)
Q[1,1] = 1
#Filling in the first row of Q
for (i in 2:11) {
  if (i %in% x) {Q[1,i] = -1} else {Q[1,i] = 1}
}
#Populate the rest of matrix by shifting each value to the right
#library(binhf)
for (i in 2:11) {
  Q[i,] = shift(Q[i-1,], places = 1, dir = "right")
}
#Prints out Q
Q
```
```

```
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11]
 [1,]    1   -1    1   -1   -1   -1    1    1    1    -1     1
 [2,]    1    1   -1    1   -1   -1   -1    1    1     1    -1
 [3,]   -1    1    1   -1    1   -1   -1   -1    1     1     1
 [4,]    1   -1    1    1   -1    1   -1   -1   -1     1     1
 [5,]    1    1   -1    1    1   -1    1   -1   -1    -1     1
 [6,]    1    1    1   -1    1    1   -1    1   -1    -1    -1
 [7,]   -1    1    1    1   -1    1    1   -1    1    -1    -1
 [8,]   -1   -1    1    1    1   -1    1    1   -1     1    -1
 [9,]   -1   -1   -1    1    1    1   -1    1    1    -1     1
[10,]    1   -1   -1   -1    1    1    1   -1    1     1    -1
[11,]   -1    1   -1   -1   -1    1    1    1   -1     1     1
```

```r
#Appending had12 and Q to create the Hadamard matrix
for (i in 2:12) {
  for (j in 2:12) {
    had12[i,j] = Q[i-1, j-1]
  }
}
#Prints the Hadamard matrix of order 12
had12
```

```
        [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
 [1,]     1    1    1    1    1    1    1    1    1     1     1     1
 [2,]    -1    1   -1    1   -1   -1   -1    1    1     1    -1     1
 [3,]    -1    1    1   -1    1   -1   -1   -1    1     1     1    -1
 [4,]    -1   -1    1    1   -1    1   -1   -1   -1     1     1     1
 [5,]    -1    1   -1    1    1   -1    1   -1   -1    -1     1     1
 [6,]    -1    1    1   -1    1    1   -1    1   -1    -1    -1     1
 [7,]    -1    1    1    1   -1    1    1   -1    1    -1    -1    -1
 [8,]    -1   -1    1    1    1   -1    1    1   -1     1    -1    -1
 [9,]    -1   -1   -1    1    1    1   -1    1    1    -1     1    -1
[10,]    -1   -1   -1   -1    1    1    1   -1    1     1    -1     1
[11,]    -1    1   -1   -1   -1    1    1    1   -1     1     1    -1
[12,]    -1   -1    1   -1   -1   -1    1    1    1    -1     1     1
```

Using Theorem 1, we are able to confirm whether or not the code worked by computing the determinant of the matrix (using the det() function in R) and then computing $n^{\frac{n}{2}}$ and checking that the absolute value of the determinant equals $n^{\frac{n}{2}}$.
Matrices formed using this construction are at the end of this paper in red.

Lemma 2 *Let m be divisible by 4 and of the form* $2^k(p+1)$*, where p is prime. Then we can construct a Hadamard matrix of order m.*

For this construction we may assume that $p \equiv 1 \pmod 4$.

Proof. Let $H = (h_{ij})$ denote a matrix of order $p + 1$. Fill in the elements of H using the following rules:

$$h_{i0} = h_{0i} = 1 \quad (1 \le i \le p),$$
$$h_{ij} = \chi(j - i) \quad (1 \le i \le p, 1 \le j \le p, i \ne j),$$
$$h_{ii} = 0 \quad (0 \le i \le p).$$

Then substitute the matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \text{and} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix},$$

respectively for $1, -1,$ and 0 in the matrix H. The resulting matrix is a Hadamard matrix of order $2^k(p + 1)$.

Below is the code to create the Hadamard matrix of order 28 using Lemma 2.

```r
```{r}
x = c()
# Finding the elements which are squares mod 13
for (i in 0:12) {
  x[i+1] = (i^2)%%13
}
x = sort(unique(x))
x = x[-(1)]
x = x + 1
Q = matrix(0,nrow=13,ncol=13)
# Constructing the first row of the Q matrix
for (i in 1:13) {
  if (i %in% x) {Q[1,i] = 1} else {Q[1,i] = -1}
}
Q[1,1] = 0
library(binhf)
# Using shift to fill in the rest of Q
for (i in 2:13) {
  Q[i,] = shift(Q[i-1,], places = 1, dir = "right")
}
# Creating the 14 x 14 matrix
B = matrix(0,nrow=14,ncol=14)
for (i in 1:14) {
  B[1,i] = 1
  B[i,1] = 1
}
for (i in 2:14) {
  for (j in 2:14) {
    B[i,j] = Q[i-1,j-1]
  }
}
B[1,1] = 0
for (i in 2:14) {
  row1 = cbind(row1,mat1)
}
had28 = matrix(0,nrow=28,ncol=28)
# Substituting in the appropriate matrices
for (i in 1:14) {
  for (j in 1:14) {
    if (B[i,j] == 1) {had28[i+(i-1),j+(j-1)] = 1}
    if (B[i,j] == 1) {had28[i+(i-1),j+j] = 1}
    if (B[i,j] == 1) {had28[i+i,j+(j-1)] = 1}
    if (B[i,j] == 1) {had28[i+i,j+j] = -1}
    if (B[i,j] == -1) {had28[i+(i-1),j+(j-1)] = -1}
    if (B[i,j] == -1) {had28[i+(i-1),j+j] = -1}
    if (B[i,j] == -1) {had28[i+i,j+(j-1)] = -1}
    if (B[i,j] == -1) {had28[i+i,j+j] = 1}
    if (B[i,j] == 0) {had28[i+(i-1),j+(j-1)] = 1}
    if (B[i,j] == 0) {had28[i+(i-1),j+j] = -1}
    if (B[i,j] == 0) {had28[i+i,j+(j-1)] = -1}
    if (B[i,j] == 0) {had28[i+i,j+j] = -1}
  }
}
# B is the Hadamard matrix of order 28
```
```

Lemma 3 *If we have a Hadamard matrix of order n_1 and a Hadamard matrix of order n_2 then we may construct a Hadamard matrix of order $n_1 n_2$.*

Paley laid out how to do this in the proof below.

Let H_1 denote the matrix of order h_1 and $-H_1$ denote the same matrix with $+1$ and -1 interchanged. Let H_2 denote the matrix of order h_2. Now for each element equal to $+1$ in H_2

we substitute the matrix H_1 and for each -1 we substitute the matrix $-H_1$. The resulting square matrix is of order $h_1 h_2$ and is orthogonal. The proof is immediate.

The method he described is the same thing as computing the Kronecker product of two matrices. Say A and B are two matrices where A is a matrix of order $m \cdot n$, then their Kronecker product $A \otimes B$ is

$$\begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \cdots & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & a_{23}B & \cdots & \cdots & a_{2n}B \\ a_{31}B & a_{32}B & a_{33}B & \cdots & \cdots & a_{3n}B \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & a_{m3}B & \cdots & \cdots & a_{mn}B \end{bmatrix}.$$

If A and B are matrices with only 1 and -1, then this is the same method that Paley described. We will now lay out a concrete example showing this computation with two Hadamard matrices of order 2 and 4.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

We replaced the three entries equal to $+1$ in A with the matrix B, and then replaced the -1 entry of A with the matrix -B just as Paley described.

Matrices of this form are at the end of the Examples section in green and white.

Hadamard Matrices of Order 52 and 92 - The Williamson Construction

An introduction to the Williamson Construction along with a list of the papers which have contributed to it over the years can be found in [3].

Theorem 4 Suppose there *exist n x n matrices A, B, C, and D that satisfy the following properties:*
- *A, B, C, and D are symmetric matrices having entries +/- 1;*
- *The matrices A, B, C, and D commute;*
- $A^2 + B^2 + C^2 + D^2 = 4nI_n.$

Then there is a Hadamard matrix of order 4n given by

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}.$$

We will now list the first row of the four 13 x 13 circulant matrices, found in [4], which lead to the construction of the Hadamard matrix of order 52.

$$A = (1, -1, -1, -1, -1, 1, -1, -1, 1, -1, -1, -1, -1)$$
$$B = (1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1)$$
$$C = (1, -1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, -1)$$
$$D = (1, -1, 1, -1, -1, 1, 1, 1, 1, -1, -1, 1, -1)$$

Below is the Hadamard matrix of order 52 using the matrices A, B, C, and D as defined above.


```
det(had52)
52^26
```

```
[1] 4.131302e+44
[1] 4.131302e+44
```

There are four matrices of order 23 which satisfy this equation. Therefore, we can use these to construct the Hadamard matrix of order 92. Leonard Baumert, S. W. Golomb, and Marshall Hall Jr. first introduced these matrices in [5]. The four matrices are circulant and below are their first rows along with the Hadamard matrix of order 92.

$$A = (1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1)$$
$$B = (1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, 1, 1, -1, 1, 1, -1)$$
$$C = (1, 1, 1, -1, -1, -1, 1, 1, -1, 1, -1, 1, 1, -1, 1, -1, 1, 1, -1, -1, -1, 1, 1)$$
$$D = (1, 1, 1, -1, 1, 1, 1, -1, 1, -1, -1, -1, -1, -1, 1, -1, 1, 1, 1, -1, 1, 1)$$


```
det(had92)
92^46
```
```
[1] 2.158937e+90
[1] 2.158937e+90
```

The code used to create these two matrices was omitted because it only consists of defining the matrices and then binding together the proper rows and columns.

Visual Library

Sylvester Matrices

Below are the Hadamard matrices of order 2, 4, 8, 16, 32, and 64 that were constructed using Sylvester's Construction.

- 2 x 2 Hadamard matrix

```
abs(det(sylv2))
2^1
```
```
[1] 2
[1] 2
```

- 4 x 4 Hadamard matrix

```
det(sylv4)
4^2
```
```
[1] 16
[1] 16
```

- 8 x 8 Hadamard matrix

```
det(sylv8)
8^4
```
```
[1] 4096
[1] 4096
```

- 16 x 16 Hadamard matrix


```
det(sylv16)
16^8
```
```
[1] 4294967296
[1] 4294967296
```

- 32 x 32 Hadamard matrix


```
det(sylv32)
32^16
```
```
[1] 1.208926e+24
[1] 1.208926e+24
```

- 64 x 64 Hadamard matrix


```
det(sylv64)
64^32
```
```
[1] 6.277102e+57
[1] 6.277102e+57
```

Paley Matrices

Below are the Hadamard matrices of order 4, 8, 12, 20, 24, 32, 44, 48, 60, 68, 72, and 80 where n = p + 1, p is prime, and p ≡ 3 (mod 4).

- 4 x 4 Hadamard matrix


```
4^(4/2)
det(had4)
` ` `
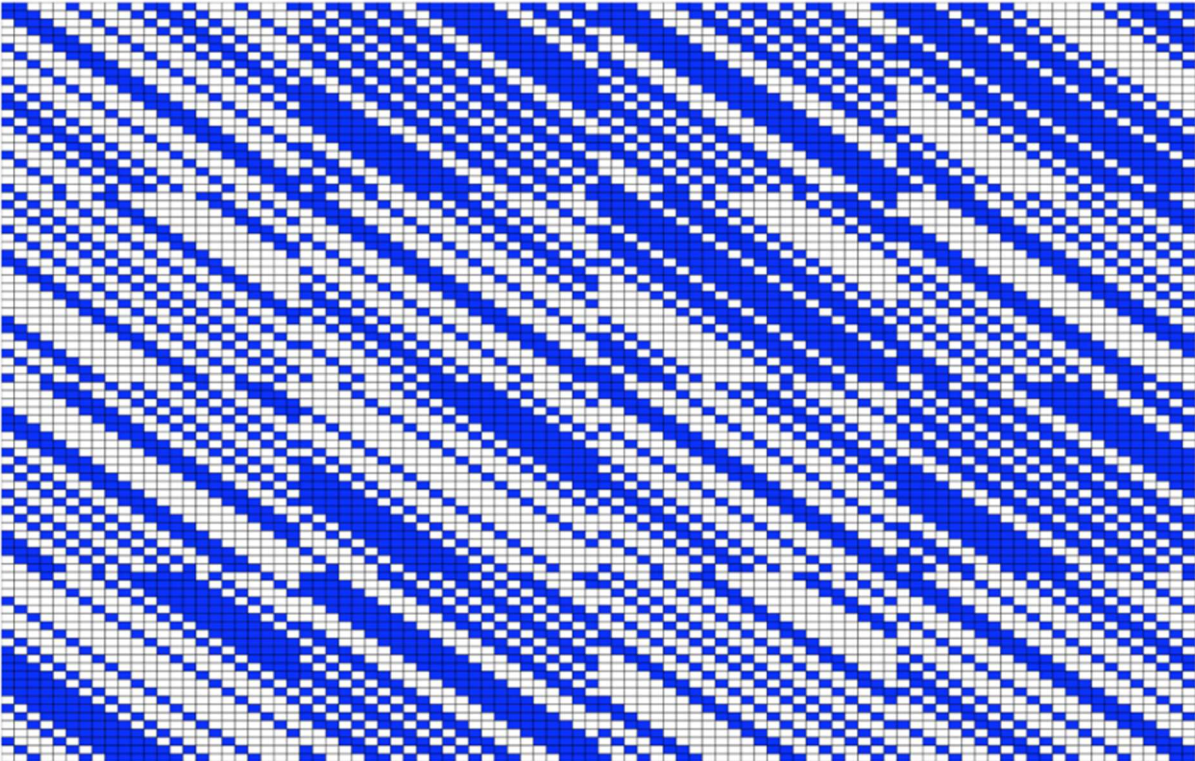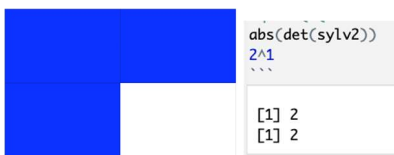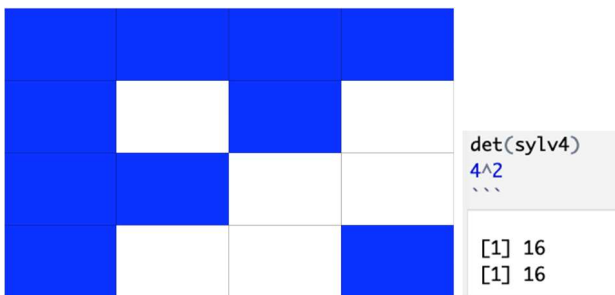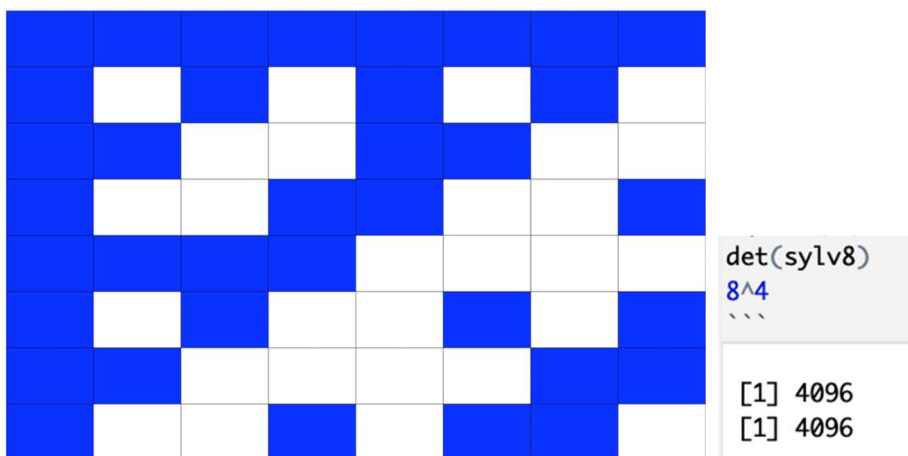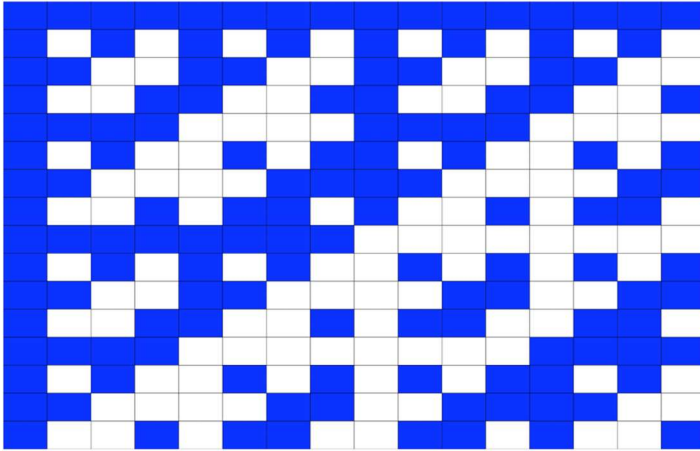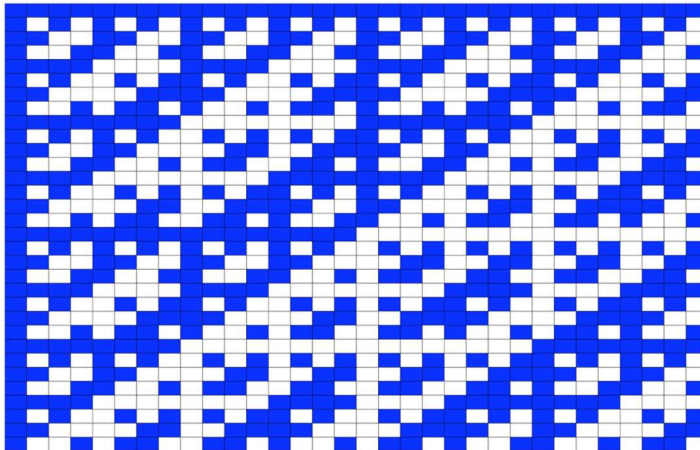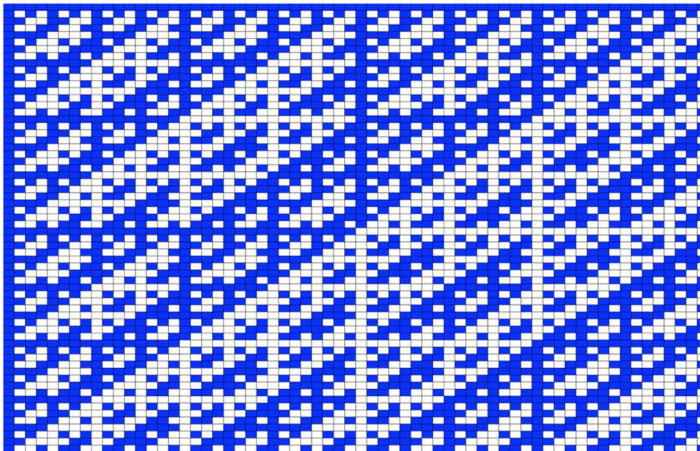
[1] 16
[1] 16
```

- 8 x 8 Hadamard matrix


```
det(had)
8^4
` ` `

[1] 4096
[1] 4096
```

- 12 x 12 Hadamard matrix


```
det(had12)
12^6
` ` `
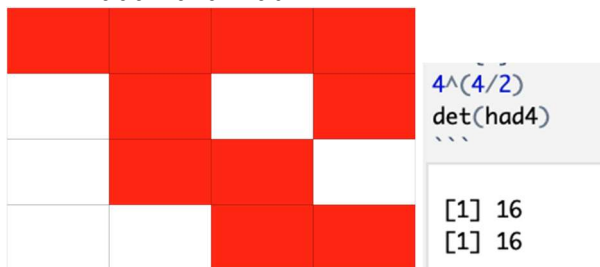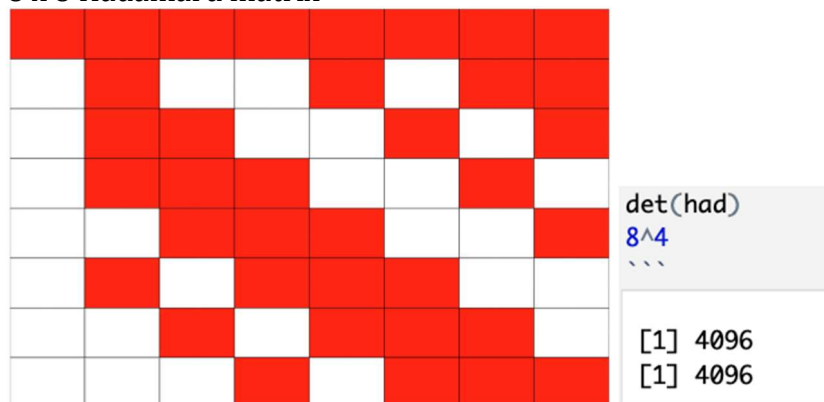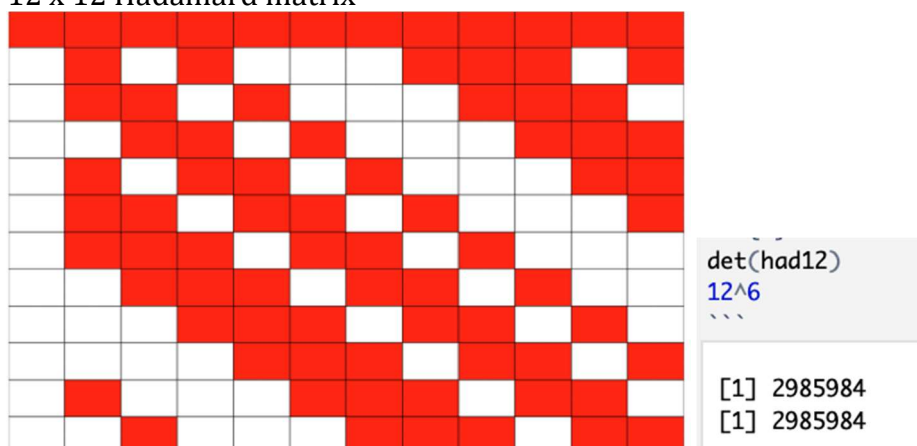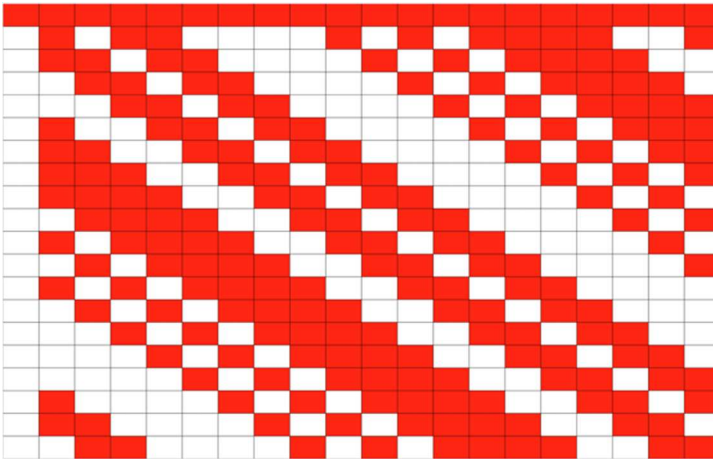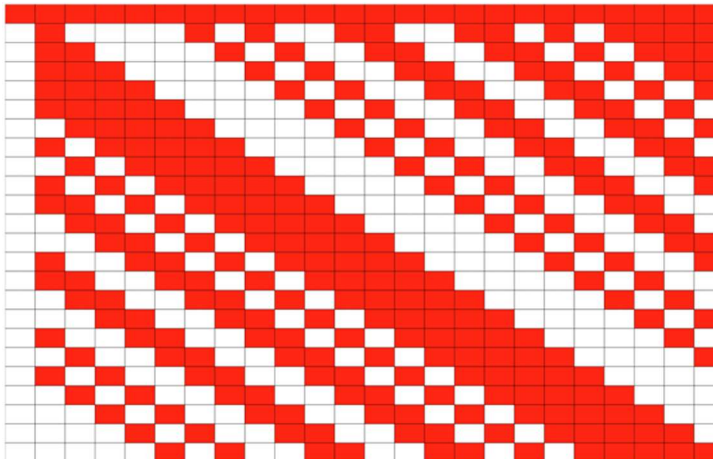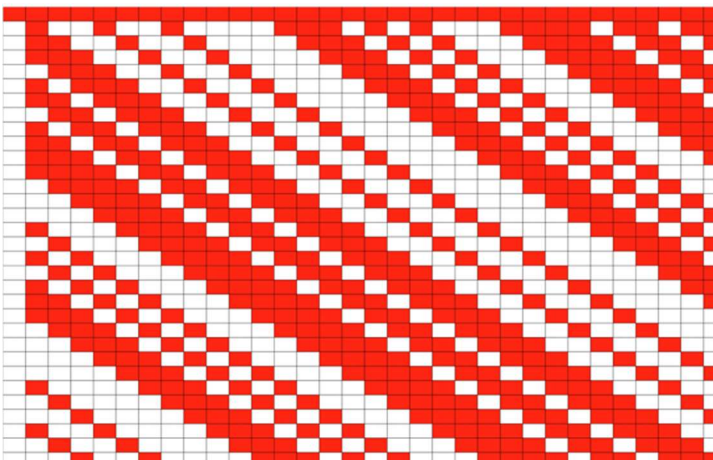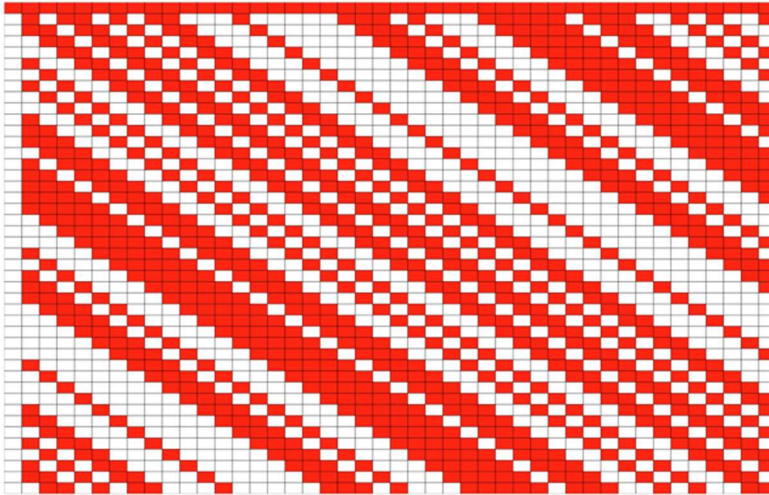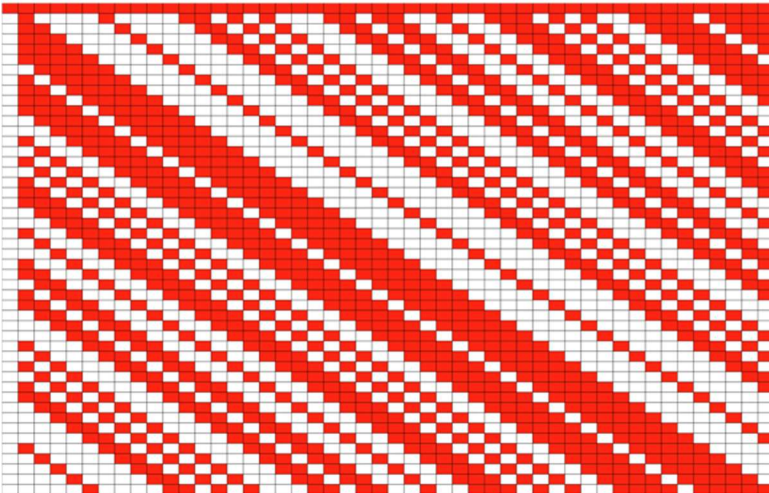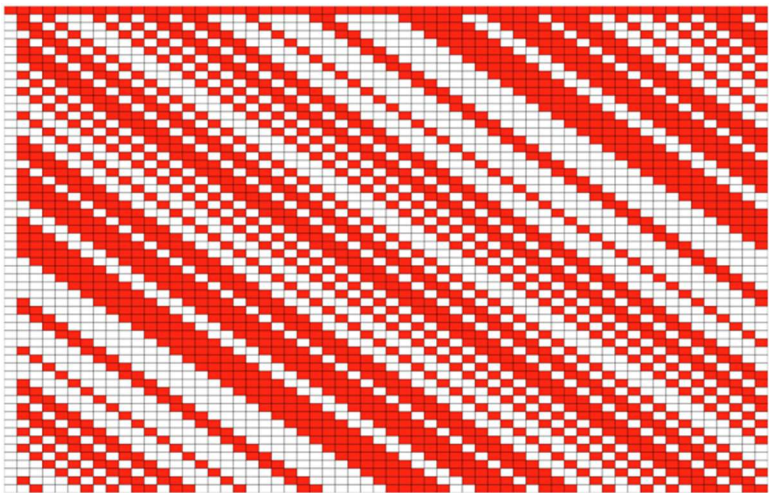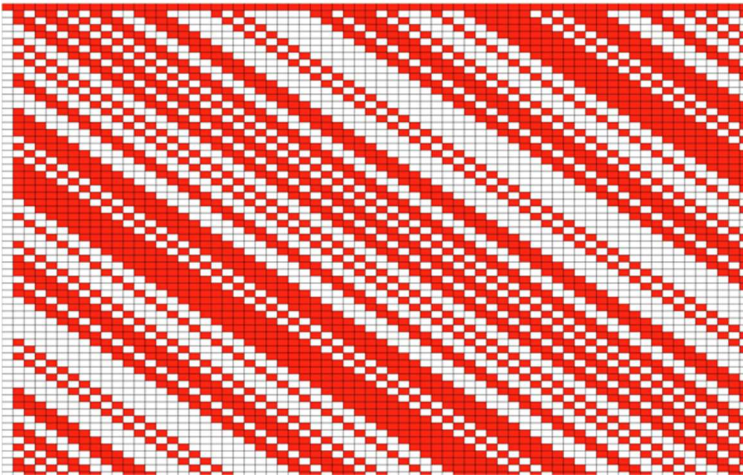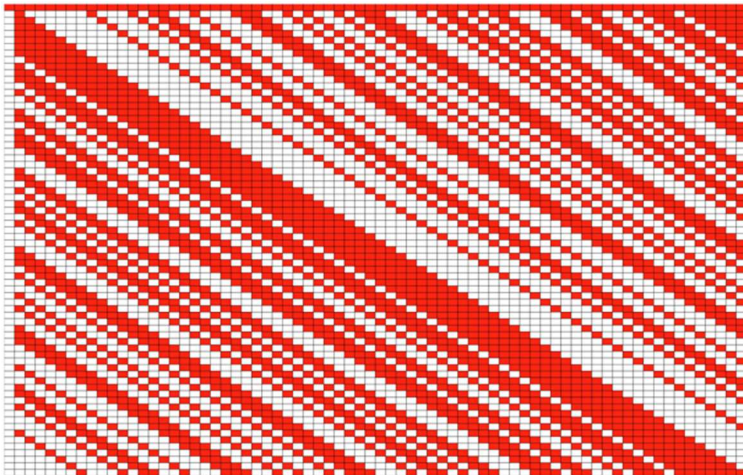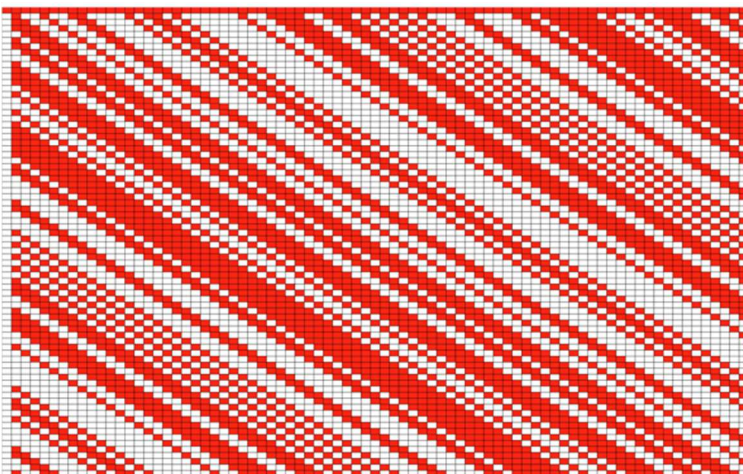
[1] 2985984
[1] 2985984
```

- 20 x 20 Hadamard matrix


```
det(had20)
20^10
```

```
[1] 1.024e+13
[1] 1.024e+13
```

- 24 x 24 Hadamard matrix


```
det(had24)
24^12
```

```
[1] 3.652035e+16
[1] 3.652035e+16
```

- 32 x 32 Hadamard matrix


```
det(had32)
32^16
```

```
[1] 1.208926e+24
[1] 1.208926e+24
```

- 44 x 44 Hadamard matrix


```
det(had44)
44^22
```

```
[1] 1.432052e+36
[1] 1.432052e+36
```

- 48 x 48 Hadamard matrix


```
det(had48)
48^24
```

```
[1] 2.237637e+40
[1] 2.237637e+40
```

- 60 x 60 Hadamard matrix


```
det(had60)
60^30
```

```
[1] 2.210739e+53
[1] 2.210739e+53
```

- 68 x 68 Hadamard matrix


```
det(had68)
68^34
```

```
[1] 2.019775e+62
[1] 2.019775e+62
```

- 72 x 72 Hadamard matrix

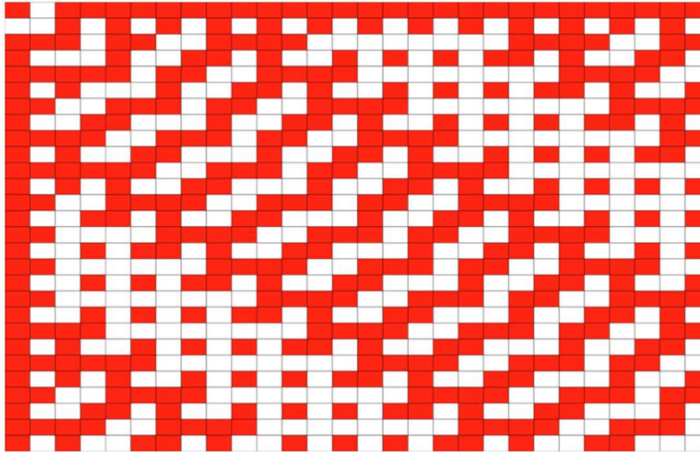```
det(had72)
72^36
```

```
[1] 7.310884e+66
[1] 7.310884e+66
```

- 80 x 80 Hadamard matrix


```
det(had80)
80^40
```

```
[1] 1.329228e+76
[1] 1.329228e+76
```

Below are the Hadamard matrices of order 28, 36, 76, and 84. These matrices are for the case where $p \equiv 1 \pmod 4$ and the order of the matrix is equal to $2(p + 1)$.
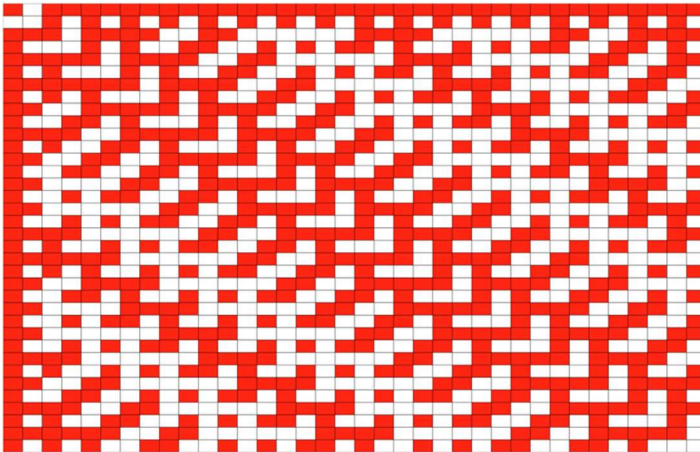
- 28 x 28 Hadamard matrix


```
28^14
det(had28)
```
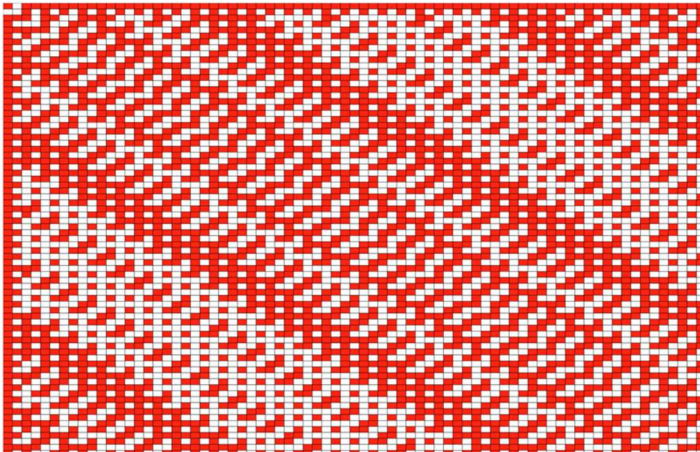
```
[1] 1.8206e+20
[1] 1.8206e+20
```

- 36 x 36 Hadamard matrix


```
36^18
det(had36)
```
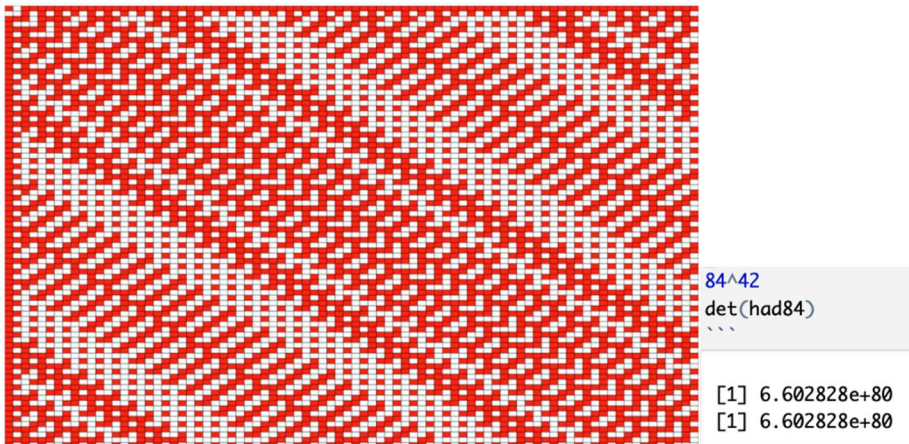
```
[1] 1.031442e+28
[1] 1.031442e+28
```
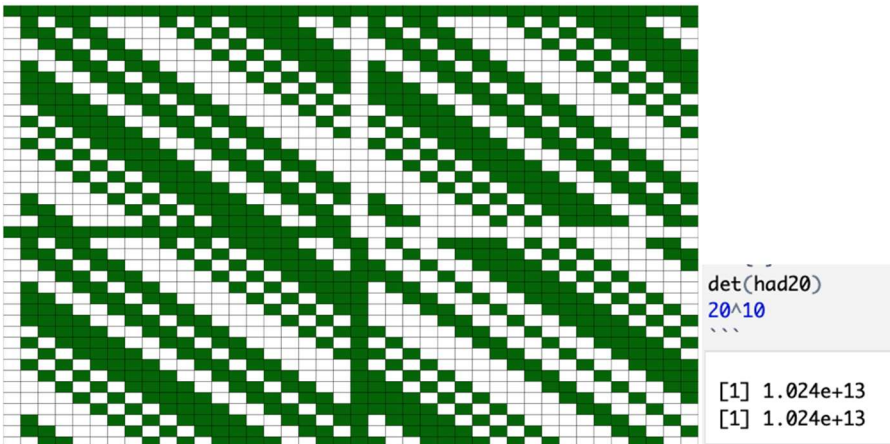
- 76 x 76 Hadamard matrix


```
76^38
det(had76)
```

```
[1] 2.957444e+71
[1] 2.957444e+71
```

- 84 x 84 Hadamard matrix


```
84^42
det(had84)
```
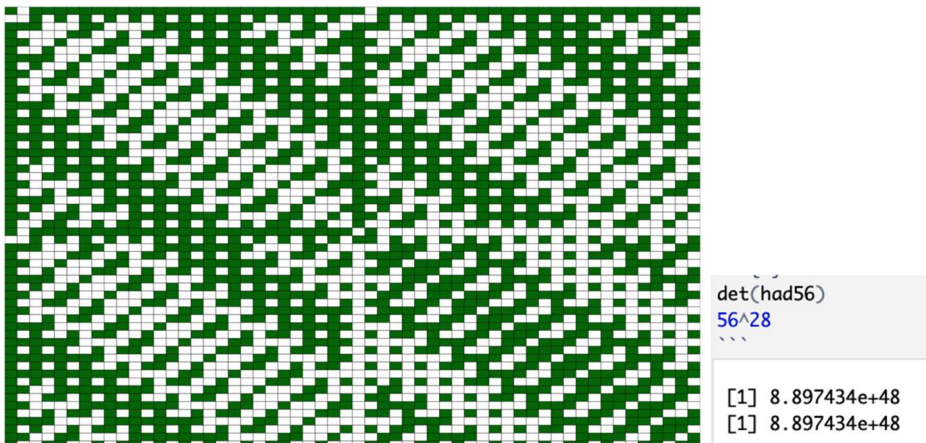```
[1] 6.602828e+80
[1] 6.602828e+80
```

Below are the Hadamard matrices of order 40, 56, 88, and 96. These were constructed using Lemma 3 with Hadamard matrices previously shown in this paper.
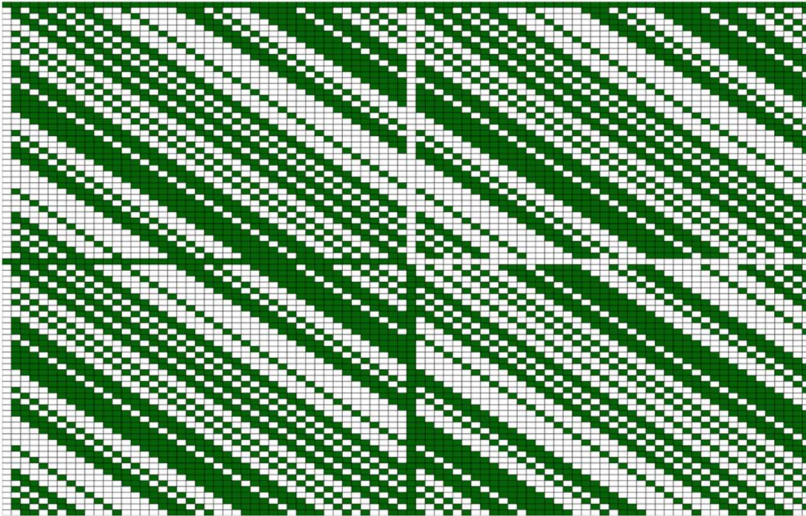
- 40 x 40 Hadamard matrix


```
det(had20)
20^10
```
```
[1] 1.024e+13
[1] 1.024e+13
```

- 56 x 56 Hadamard matrix


```
det(had56)
56^28
```
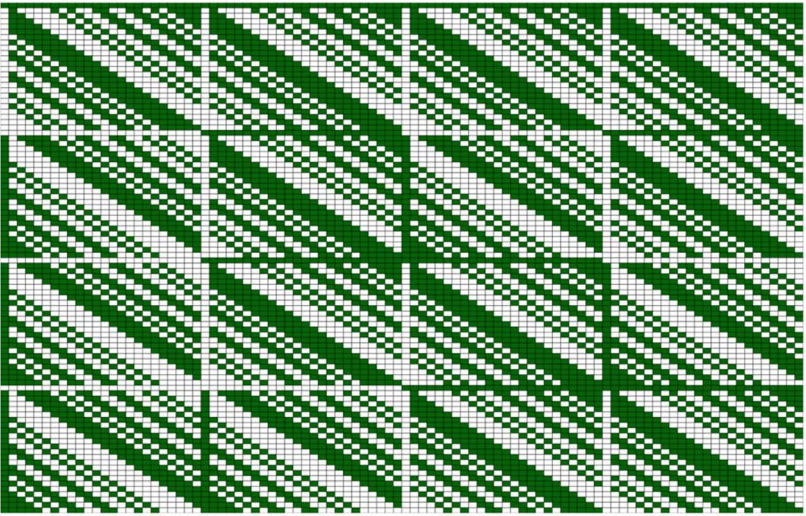```
[1] 8.897434e+48
[1] 8.897434e+48
```

- 88 x 88 Hadamard matrix



```
det(had88)
88^44
```
```
[1] 3.607759e+85
[1] 3.607759e+85
```

- 96 x 96 Hadamard matrix



```
det(had96)
96^48
```
```
[1] 1.409351e+95
[1] 1.409351e+95
```

# REFERENCES

[1] Cameron, Peter. 2006. "Hadamard Matrices." *The Encyclopedia of Design Theory* .

[2] Paley, R. E. A. C. 1933. "On Orthogonal Matrices ." *Journal of Mathematics and Physics Volume 4* 311-320.

[3] A. Hedayat, W.D. Wallis. 1978. "Hadamard Matrices and their Applications." *The Annals of Statistics Volume 6* 1184-1238.

[4] Seberry, Jennifer. 1999. *Williamson Matrices* . November 1. Accessed April 10, 2019. https://www.uow.edu.au/~jennie/WILLIAMSON/williamson.html.

[5] Leonard Baumert, S. W. Golomb, Marshall Hall Jr. 1962. "Discovery of an Hadamard Matrix of Order 92 ." *Bull. Amer. Math. Soc. Volume 68* 237-238.